Newton's Method for Finding Roots

An introduction

In this course and in previous math courses you have spent a lot of time learning how to solve equations such as $x^2-2x+8=0$, $x^4+6x^2=-9$, and $100e^{2t}=800$. A sad fact of life is that the techniques that you learned, no matter how useful, were "tricks" that work in a relatively small set of circumstances and not otherwise. Most equations cannot, even in principle, be solved in closed form. Take, for example, $x^2=\exp(x)$. There is no way to "isolate" the x; to get x on one side of the equation and a number or set of numbers on the other side. Such equations can be solved only approximately. There are many techniques available. One of the best is "Newton's method." It uses the idea that the tangent line to the curve of a differentiable function closely approximates the curve near the point of tangency.

Note that any equation can be re-written in the form $f(x)=0$. For instance, the equation $\exp(2x)=x^2-4$ can be rearranged to get

$$\exp(2x)-x^2+4=0.$$

In this case, $f(x)=\exp(2x)-x^2+4$. Thus solving equations can, alternatively, be thought of as finding "roots" of functions, that is, places where the graph of the function crosses the x-axis. Newton's method gives us a way of approximating roots of functions.

Suppose we have a function f, and we want to solve $f(x)=0$. To use Newton's method, you must have an initial guess x_0. The next guess, x_1, is found at the intersection of the x-axis with the tangent line to $y=f(x)$ at $(x_0,f(x_0))$, as shown in the figure below.

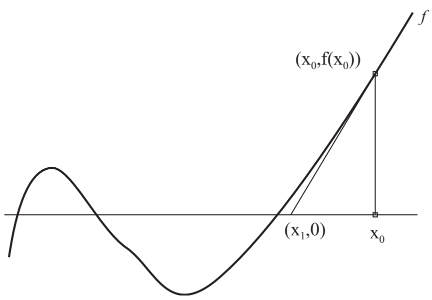


Figure 1: We use our initial guess, x_0, to find out next guess, x_1.

I. Finding the Recursion Relation

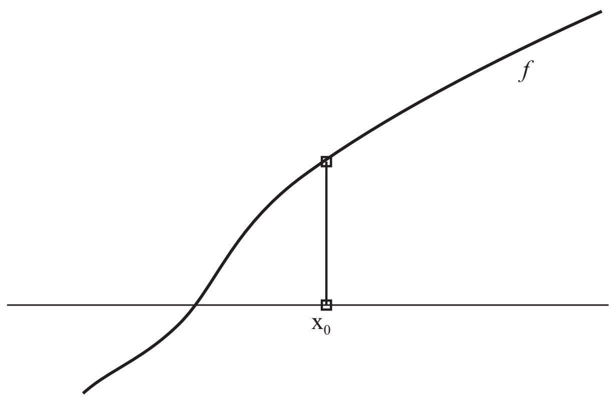
Given x_0, we need to find the formula for x_1. Take a good look at the figure before you proceed. Think about how you might you go about finding x_1 if you know x_0 and f.

1. Look at the graph shown in Figure 1. Find two ways of computing the slope of the line joining $(x_0, f(x_0))$ and $(x_1, 0)$. (Hint: one method is algebraic and the other requires calculus!)

2. These two expressions for the slope are the same. So use the fact that they are equal to show that
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}.$$

Once we have x_1, we repeat the process—taking x_1 as our initial guess— to get x_2 from x_1, then x_3 from x_2, and so on. If all goes well, the x_i's get closer and closer to the root we are seeking.

3. On the figure below draw in x_1, x_2, and x_3. Notice how each successive guess gets closer to the actual root of the function.



4. Write a formula for x_2 in terms of x_1.

5. Write a formula for x_3 in terms of x_2.

6. Use your answers to the previous problems to help you find a pattern that you can generalize to get a formula for x_{n+1} in terms of x_n.

   This formula is called a **recursion relation** because it is used over and over again with successive guesses.